

CHAPTER 10. THE BIVARIATE NORMAL DISTRIBUTION



10.1. DEFINITIONS

10.1.1. Bivariate Normal Distribution

Let X and Y be two random variables with joint probability density function $f(x, y)$. If $f(x, y)$ is of the form

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_x}{\sigma_x} + \rho\frac{y-\mu_y}{\sigma_y}\right]^2 - \frac{\rho^2}{2(1-\rho^2)}\left[\frac{y-\mu_y}{\sigma_y}\right]^2 - \frac{\rho^2}{2(1-\rho^2)}\left[\frac{x-\mu_x}{\sigma_x}\right]^2\right\}$$

where $\mu_x, \mu_y, \sigma_x, \sigma_y$ are constants and $|\rho| < 1$, then X and Y are said to have a bivariate normal distribution.

The marginal probability density functions of X and Y are given by

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{x-\mu_x}{\sigma_x}\right]^2\right\}$$

$$f_Y(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{y-\mu_y}{\sigma_y}\right]^2\right\}$$

10.1.2. Conditional Distributions

The conditional probability density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_x}{\sigma_x} + \rho\frac{y-\mu_y}{\sigma_y}\right]^2 - \frac{\rho^2}{2(1-\rho^2)}\left[\frac{y-\mu_y}{\sigma_y}\right]^2\right\}$$

The conditional probability density function of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sigma_y\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_y}{\sigma_y} + \rho\frac{x-\mu_x}{\sigma_x}\right]^2 - \frac{\rho^2}{2(1-\rho^2)}\left[\frac{x-\mu_x}{\sigma_x}\right]^2\right\}$$

The conditional means and conditional variances of X and Y are given by

$$\mu_{X|Y} = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y) \quad \sigma_{X|Y}^2 = \sigma_x^2(1 - \rho^2)$$

$$\mu_{Y|X} = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x) \quad \sigma_{Y|X}^2 = \sigma_y^2(1 - \rho^2)$$

10.2. THE BIVARIATE NORMAL DISTRIBUTION

Let X and Y be two random variables with joint probability density function $f(x, y)$. If $f(x, y)$ is of the form